Recall: wave equ. for disk

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$

considered simpler case of radial symmetry

$$\Rightarrow \quad u(r,t) = f(r)\, h(t)$$

f satisfied ODE

$$\boxed{r \frac{d}{dr}\left(r \frac{df}{dr}\right) + (\lambda r - m^2) f = 0}$$

general solution

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r)\left(a_n \cos \sqrt{\lambda_{0n}}\, c t + b_n \sin \sqrt{\lambda_{0n}}\, c t\right)}$$

? how to calculate coefficients a_n and b_n from initial conditions?

Problem can be solved (in principle) for the following general set up (Sturm - Liouville Problems
See Sections 5.3 , 5.5)

Consider ODE

$$\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\,\phi + \lambda\,\sigma(x)\,\phi = 0}$$

here $p(x)$, $q(x)$, $\sigma(x)$ are given functions

(e.g. from previous page (after dividing by r)

$$\frac{d}{dr}\Big(\underbrace{r}_{p(r)}\frac{df}{dr}\Big) \underbrace{- \frac{m^2}{r}}_{q(r)} f + \lambda \underbrace{r}_{\sigma(r)} f = 0$$)

boundary cond. later.

notation: write $L(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\,u$

Crucial observation

Lagrange Identity (differential form)

$$\boxed{u\,L(v) - v\,L(u) = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)}$$

proof. $u\,L(v) - v\,L(u) = u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + \cancel{q(x)\,uv}$

$$- \left[v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \cancel{q(x)\,vu} \right]$$

$$\boxed{u\,L(v) - v\,L(u) = u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)}$$

Recall product rule $(fg)' = f'g + fg'$

$$\Rightarrow fg' = (fg)' - f'g$$

apply for $f = u$ $\quad g = p(x)\frac{dv}{dx}$

and for $f = v$ $\quad g = p(x)\frac{du}{dx}$

$$\Rightarrow uL(v) - vL(u) = \frac{d}{dx}\left(u\,p(x)\frac{dv}{dx}\right) - \cancel{\frac{du}{dx}\,p(x)\,\frac{dv}{dx}}$$

$$- \left[\frac{d}{dx}\left(v\,p(x)\frac{du}{dx}\right) - \cancel{\frac{dv}{dx}\,p(x)\,\frac{du}{dx}}\right]$$

$$= \frac{d}{dx}\left(u\,p(x)\frac{dv}{dx} - v\,p(x)\frac{du}{dx}\right)$$

Integrate Lagrange identity (diff form)

$$\Rightarrow \boxed{\int_a^b u L(v) - v L(u)\, dx = p\left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\Bigg|_{x=a}^{x=b}}$$

apply this to our ODE:

$$\boxed{L(f) + \lambda \sigma(x) f = 0}$$

with boundary conditions

$$\boxed{\begin{aligned} \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) &= 0 \\ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) &= 0 \end{aligned}}$$

Observe: we have considered special cases.

$\beta_1 = \beta_3 = 1 \quad \beta_2 = \beta_4 = 0$
i.e. $\phi(a) = 0 = \phi(b)$.

or

$\beta_2 = \beta_4 = 1, \quad \beta_1 = \beta_3 = 0$
i.e. $\phi'(a) = 0 = \phi'(b)$

Def ϕ is a generalized eigenfunction with eigenvalue λ

if $L(\phi) + \lambda \sigma(x) \phi = 0$

Theorem let ϕ_m and ϕ_n be eigenfunctions with eigenvalues λ_m and λ_n

$$\Longrightarrow \boxed{\int_a^b \phi_m(x) \phi_n(x) \sigma(x)\, dx = 0 \quad \text{if } \lambda_m \neq \lambda_n}$$

proof Calculate $\int_a^b \phi_m L(\phi_n) - \phi_n L(\phi_m) dx$ in 2 different ways.

ⓐ $= \int_a^b \phi_m(-\lambda_n \sigma(x) \phi_n) + \phi_n(+\lambda_m \sigma(x) \phi_m) dx$

$$= \boxed{(\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x)\, dx.}$$

(b) $= p(x)\left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx}\right)\Big|_{x=a}^{x=b}$

use Lagrange identity

observe: both ϕ_m and ϕ_n satisfy the same boundary conditions

....

$\Rightarrow \boxed{\text{r.h.s.} = 0}$

(just check for simple cases $\phi(a) = 0 = \phi(b)$

or $\phi'(a) = 0 = \phi'(b)$)

get back to our example:

$$\underbrace{\frac{d}{dr}\left(r\frac{df}{dr}\right)+\frac{m^2}{r}f}_{L(f)} - \lambda r f = 0$$

$\sigma(r) = r$

caveat: • $\frac{m^2}{r}$ has a pole at $r=0$

• explicit boundary cond. only at $r=a$.

implicit " " $|f(0)|<\infty$

Nevertheless, formula of last theorem still holds also for this case.

Recall: $\phi_n(r) = J_0(\sqrt{\lambda_{0n}}\, r)$

is an eigenfunction of $\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r} f + \lambda r f = 0$

with eigenvalue $\lambda_{0n} = \frac{z_{0n}}{a^2}$

$\sigma(r) = r$

$$\Rightarrow \boxed{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)\, J_0(\sqrt{\lambda_{0k}}\, r)\, r\, dr = 0 \quad \text{if } n \neq k}$$

apply this to calculate coefficients in series solution
subject to boundary cond

$$\boxed{\begin{aligned} u(r,0) &= \alpha(r) \\ \frac{\partial u}{\partial t}(r,0) &= \beta(r) \end{aligned}}$$

$$\boxed{\begin{aligned} \alpha(r) &= \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r)\, a_n = u(r,0) \\ \beta(r) &= \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r)\, b_n \sqrt{\lambda_{0n}}\, c \end{aligned}}$$

$$\Rightarrow \boxed{a_n = \frac{\int_0^a \alpha(r)\, J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr}{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)^2\, r\, dr}}$$

$$\boxed{b_n \sqrt{\lambda_{0n}}\, c = \frac{\int_0^a \beta(r)\, J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr}{\int_0^b J_0(\sqrt{\lambda_{0n}}\, r)^2\, r\, dr}}$$